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Exactly solvable model of interface growth

T. J. Newman

Department of Physics, University of Oxford, Theoretical Physics, 1 Keble Road, OX1 3NP, United Kingdom

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We propose and exactly solve a model of interface growth. The model is applicable to Kardar-Parisi-Zhang type interfaces growing into an environment whose density decreases exponentially with height. We find that the average height of the interface grows as $\ln(t)$ for all spatial dimensions d. The interface width has a much richer dependence on d, showing a nontrivial crossover behavior around $d = 2$.

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There has been much interest in recent years in the nonequilibrium dynamics of interfaces [1]. Despite great effort, there is a lack of rigorous analytic treatments of the various proposed models. Most of our understanding of the kinetics of interfaces comes from the wide range of large-scale computer simulations that have been performed. Making the connection between the simulation results and the current theoretical models has often proved difficult. Perhaps the most popular theoretical model is due to Kardar, Parisi, and Zhang (KPZ) [2], who proposed a simple nonlinear Langevin equation for the dynamics of the interface. A dynamic renormalization group (RG) study of this equation indicates the possibility of strong-coupling (SC) behavior for (substrate) dimension $d \geq 2$ (which seems from computer simulation to be the relevant fixed point [3])—unfortunately the evaluation of exponents at this SC fixed point is yet to be achieved due to the intrinsic difficulty in working in the nonperturbative regime. In this Rapid Communication we shall present a model which bears some relation to the original KPZ model, and has the feature of being exactly solvable. It is to be admitted that the solvability of the model is its main motivation, although we shall briefly describe a possible physical application. We shall find that the interface width $W(t)$ exhibits a nontrivial crossover around $d = 2$. Before stating the results we shall first introduce and discuss the model.

We consider the following equation of motion for the interface height $h(\mathbf{x}, t)$:

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \frac{2\nu}{\lambda}\eta \exp(-\lambda h/2\nu), \qquad (1)$$

where $\eta(\mathbf{x}, t)$ is a stochastic source whose distribution will be defined below. This equation is precisely the KPZ equation except for the multiplicative factor in the noise term. The first thing to note is that this equation is no longer in terms of the relative height (i.e., the deviation from the average height of the interface) since h appears explicitly in the equation without being acted upon by some differential operator. We interpret h as an *absolute* height measured in some fixed reference frame. We can then think of the new multiplicative noise in terms of a density gradient in the environment (the vapor, for instance, in a solid-on-solid interface system) into which the interface is growing—the density of the environment decaying exponentially with increasing height. On physical grounds we may foresee that the average height of the interface will grow as $\ln(t)$ since this then conserves the rate of deposition. We may also expect the interface fluctuations to decay for large times due to the weakening of the noise as h increases. We shall see that both of these ideas are correct—therefore the *physical* contact with the original KPZ equation and related models is lost (since there one expects fluctuations to increase with time).

Before describing the solution of Eq. (1) we first state the results for the asymptotic behavior of the interface width:

$$W(t) \sim \begin{cases} \alpha(d)\dfrac{\nu^{1-d/4}}{\lambda} t^{-(d+2)/4}, & 0 < d < 2 \\ \left(\dfrac{\nu}{\pi\lambda^2}\right)^{1/2} t^{-1}(\ln t)^{1/2}, & d = 2 \\ \beta(d)\dfrac{\nu^{1/2}}{\lambda} t^{-(d+2)/2d}, & d > 2 \end{cases} \qquad (2)$$

where $\alpha(d)$ and $\beta(d)$ are constants.

The above results indicate that this model has a critical dimension of $d = 2$ which separates two different "fixed point" structures (in the RG sense). It is tempting to draw some general correspondence between this and the postulated fixed point structure for the KPZ model; however, we shall desist from this due to the qualitatively different physics of the two models. To what extent the above results may be described in RG language (for instance, whether the results for $d > 2$ correspond to a strong-coupling fixed point) can only be answered by performing an RG analysis on Eq. (1) which seems nontrivial due to the lack of a simple "bare" theory.

The model defined above was in fact chosen for study because it may be linearized directly with the Hopf-Cole transformation $w = \exp(\lambda h/2\nu)$, yielding a linear diffusion equation for w:

$$\partial_t w = \nu \nabla^2 w + \eta. \qquad (3)$$

The same transformation may be applied to the KPZ equation and one obtains a linear diffusion equation in w, but with multiplicative noise—this equation may be interpreted as describing the evolution of the generating function for directed walks in a random medium [4]. There is no such analogy to directed walks in the present model.

The solution of Eq. (3) is easily obtained in terms of the heat kernel $g(\mathbf{x}, t) = (4\pi\nu t)^{-d/2}\exp(-x^2/4\nu t)$ and we then have the exact solution of Eq. (1) as

$$h(\mathbf{x},t) = (2\nu/\lambda)\ln\left\{1 + \int d^d y \int_0^t dt' g(\mathbf{x}-\mathbf{y},t-t')\eta(\mathbf{y},t')\right\}, \tag{4}$$

where, for simplicity, we have chosen the initial condition to be that of a flat interface with $h(\mathbf{x},0) = 0$, although more general initial conditions may be handled within this approach. We mention that one may obtain an exact solution to Eq. (1) with the inclusion of an arbitrary function of time $f(t)$ appearing as a prefactor in the multiplicative noise term, by using the same transformation defined above. [The case of $f(t) = e^{ct}$ is interesting since then the average height of the interface grows *linearly* in time with velocity $v = 2\nu c/\lambda$. The fluctuations in this case will be considered in a longer paper [5].]

From the above solution for h we see that the parameter λ only appears in the prefactor. Therefore there is no weak-coupling perturbation theory (in powers of λ) available with which to study this model in contrast to the original KPZ equation where the bare theory ($\lambda = 0$) is the soluble Edwards-Wilkinson (EW) model [6]. It is clear from Eq. (4) that the calculation of quantities such as the average height and the average interface width will require averages to be performed over logarithms of the noise η. As in a previous work [7] we choose to make use of the following representation of the logarithm function:

$$\ln z = \int_0^\infty \frac{du}{u}(e^{-u} - e^{-uz}). \tag{5}$$

This representation is useful since the argument of the logarithm in Eq. (4) is a space-time integral and the use of Eq. (5) enables us to average over the *exponential* of the space-time integral which is a very natural calculational step. So, combining Eq. (4) and Eq. (5) we have

$$h(\mathbf{x},t) = (2\nu/\lambda)\int_0^\infty \frac{du}{u}e^{-u}[1-\psi(u)], \tag{6}$$

where

$$\psi(u) = \exp\left(-u\int d^d y \int_0^t dt' g(\mathbf{x}-\mathbf{y},t-t')\eta(\mathbf{y},t')\right). \tag{7}$$

Before proceeding with the calculation of averaged quantities we must specify the noise distribution. From Eq. (1) one can see that for negative values of the field h, the noise becomes exponentially large—on physical grounds this is to be avoided. We therefore choose a noise distribution which provides only positive values of η so that the field h is always positive. In this paper we shall choose the distribution to be uncorrelated and of Poisson type (with $\eta \in [0,\infty]$):

$$P[\eta] \sim \exp\left(-(1/D)\int d^d y \int_0^\infty dt\, \eta(\mathbf{y},t)\right). \tag{8}$$

Henceforth all averages over P will be denoted by angled brackets. Other distributions will be considered in a following paper [5].

Since this Rapid Communication is primarily concerned with results we shall not describe the details of the calculation in much depth (a more detailed account will be presented in [5]). In order to evaluate the average height one averages Eq. (6) using the above distribution. This averaging necessitates the introduction of spatial and temporal "lattice" cutoffs, which are then absorbed by rescaling all lengths and times with respect to these cutoffs. For the function $\langle\psi(u)\rangle$ one finds

$$\langle\psi(u)\rangle = \exp[-uDt\phi(uD/\tau)], \tag{9}$$

where $\tau = (4\pi\nu t)^{d/2}$ and

$$\phi(z) = [z\Gamma(d/2+2)]^{-1} \times \int_0^\infty dy\, y^{d/2} F(1, 1+2/d; 2+2/d; -e^y/z), \tag{10}$$

where $F(a,b;c;x)$ is the hypergeometric function [8].

To calculate the asymptotic form of the average height one needs only the first term in the small-u expansion of ϕ, which to the order required is simply given by unity. One then sees from Eq. (6) and Eq. (9) that the average height $\langle h\rangle \sim \ln(t)$ for all spatial dimensions d. This result may also be used to evaluate the evolution of the local interface roughness $E(t) = \langle(\boldsymbol{\nabla} h)^2\rangle$. From Eq. (1) we expect $E(t) \sim \partial_t\langle h\rangle$. This implies $E(t) \sim 1/t$ for all d.

The evaluation of the width is more involved since it requires the next order in the small-u expansion of ϕ—this next to leading term will be found to be strongly dependent on d. We define the width by $W(t) = [\langle h^2\rangle - \langle h\rangle^2]^{1/2}$. By double application of the logarithm representation (5) one finds the following form:

$$W^2(t) = (2\nu/\lambda)^2 \int_0^\infty \frac{du}{u}\int_0^\infty \frac{dv}{v} e^{-(u+v)}[\langle\psi(u+v)\rangle - \langle\psi(u)\rangle\langle\psi(v)\rangle]. \tag{11}$$

The leading terms in the small-u expansion of ϕ are found to be ($z = uD/\tau$)

$$\phi(z) = \begin{cases} 1 - \frac{1}{(2-d)2^{d/2}}z + O(z^2, z^{2/d}), & 0 < d < 2 \\ 1 + (z/4)(\ln(z) - 1) + O(z^2), & d = 2 \\ 1 - \frac{2\pi/d}{\sin(2\pi/d)}\left(\frac{d}{d+2}\right)^{1+d/2} z^{2/d} + O(z), & d > 2. \end{cases} \tag{12}$$

Substituting the above forms for ϕ into Eq. (9) and using Eq. (11) then yields the asymptotic results given earlier in Eq. (2) [10] (the d-dependent prefactors for the width are complicated and will be given in [5]). This completes our sketch of the analytic solution for the asymptotic behavior of the model defined in Eq. (1).

There are many possible extensions to this work. Firstly, one can try to use this method to calculate the general correlation function $C(\mathbf{r},t) = \langle[h(\mathbf{r},t) - h(\mathbf{0},t)]^2\rangle$. Knowledge of this function would be essential if one is to prove the existence of scaling in this model. Also, one may repeat the above calculation for different choices of the distribution $P[\eta]$—one expects a fair degree of insensitivity of the exponents to the exact choice of P, although similar examples are known where the exponents are dependent upon the gross characteristics of the distribution [7,9]. The evaluation of the short-time behavior of the system may also be of interest. We should also point out that the physical importance of the nonlinear term $(\nabla h)^2$ in Eq. (1) is not clear. It is essential in order to achieve an exact solution, but would appear to be subdominant to the diffusion term when the interface fluctuations are small. It would be of interest to clarify this through some approximate analysis of an equation similar to Eq. (1), but with arbitrary coefficients in the noise term (so that λ may be taken to approach zero.) It may be that the nonlinearity is important only for $d > 2$ where the interface fluctuations for the EW-type model (i.e., with no nonlinear term) are already negligible.

One hopes that the exact results obtained from this model will be of some use in the broader picture of interface growth (and associated areas such as fluid turbulence, reaction-diffusion models, etc.). This model may perhaps be used as a testing ground for the more sophisticated techniques used to study these systems, such as mode-coupling theories and the dynamic renormalization group.

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[1] For general reviews see J. Krug and H. Spohn, in *Solids Far From Equilibrium: Growth, Morphology and Defects*, edited by C. Godreche (Cambridge University Press, Cambridge, England, 1991); *Dynamics of Fractal Surfaces*, edited by F. Family and T. Vicsek (World Scientific, Singapore, 1991).

[2] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[3] For example, see J. M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. **62**, 2289 (1989).

[4] M. Kardar, Phys. Rev. Lett. **55**, 2235 (1985); A. J. McKane and M. A. Moore, *ibid.* **60**, 527 (1988).

[5] T. J. Newman (unpublished).

[6] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London Ser. A **381**, 17 (1982).

[7] S. E. Esipov and T. J. Newman, Phys. Rev. E **48**, 1046 (1993).

[8] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, 10th ed. (Dover, New York, 1972).

[9] S. E. Esipov, Phys. Rev. E **49**, 2070 (1994).

[10] The results displayed in Eq. (2) are valid in the limit of vanishing temporal cutoff. The case of nonzero cutoff is planned to be examined in Ref. [5].